A remark on the uniqueness of positive solutions to semilinear elliptic equations with double power nonlinearities

Shinji Kawano
Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan

Abstract

We consider the uniqueness of positive solutions to

$$\begin{cases} \triangle u - \omega u + u^p - u^{2p-1} = 0 \quad \text{in } \mathbb{R}^n, \\ \lim\limits_{|x|\to\infty} u(x) = 0. \end{cases} \tag{1}$$

It is known that for fixed $p > 1$, a positive solution to (1) exists if and only if $\omega \in (0, \omega_p)$, where $\omega_p := \dfrac{p}{(p+1)^2}$. We deduce the uniqueness in the case where ω is close to ω_p, from the argument in the classical paper by Peletier and Serrin [9], thereby recovering a part of the uniqueness result of Ouyang and Shi [8] for all $\omega \in (0, \omega_p)$.

1 Introduction

We shall consider a boundary value problem

$$\begin{cases} u_{rr} + \dfrac{n-1}{r} u_r - \omega u + u^p - u^{2p-1} = 0 \quad \text{for } r > 0, \\ u_r(0) = 0, \\ \lim\limits_{r\to\infty} u(r) = 0, \end{cases} \tag{2}$$

where $n \in \mathbb{N}$, $p > 1$ and $\omega > 0$. The above problem arises in the study of

$$\begin{cases} \triangle u - \omega u + u^p - u^{2p-1} = 0 \quad \text{in } \mathbb{R}^n, \\ \lim\limits_{|x|\to\infty} u(x) = 0. \end{cases} \tag{3}$$

Indeed, the classical work of Gidas, Ni and Nirenberg [4, 5] tells us that any positive solution to (3) is radially symmetric. On the other hand, for a solution $u(r)$ of (2), $v(x) := u(|x|)$ is a solution to (3).

The condition to assure the existence of positive solutions to (3) (and so (2)) was given by Berestycki and Lions [1] and Berestycki, Lions and Peletier [2]: A solution to (2) with fixed $p > 1$ exists if and only if $\omega \in (0, \omega_p)$, where

$$\omega_p = \frac{p}{(p+1)^2}.$$

We shall review what this ω_p is for in Section 2. Throughout this paper, *a solution* means a classical solution.

Uniqueness of positive solutions to (2) had long remained unknown. Finally in 1998 Ouyang and Shi [8] proved uniqueness for (2) with all $\omega \in (0, \omega_p)$, $p > 1$. See also Kwong and Zhang [6].

In this present paper, we prove that for ω close to ω_p, the uniqueness result is obtained directly from the classial result given by Peletier and Serrin [9] in 1983. For another attempt to obtain the uniqueness when ω is close to ω_p, see Mizumachi [7]. Our result of the present paper is the following:

Theorem 1. *Let $n \in \mathbb{N}$, $p > 1$ and $\omega \in [a_p, \omega_p)$, where* $a_p := \dfrac{p(7p-5)}{4(p+1)(2p-1)^2}$. *Then* (2) *has exactly one positive solution.*

Remark 1. Note that

$$0 < a_p < \omega_p = \frac{p}{(p+1)^2}, \qquad p > 1.$$

In the next section we clarify the definitions of ω_p and a_p from the point of view from [9].

2 Study of the nonlinearity as a function

In this section, we study the properties of the function $f(u) := -\omega u + u^p - u^{2p-1}$ in $(0, \infty)$, where $\omega > 0$ and $p > 1$ are given constants.

First we define $F(u) := \int_0^u f_{\omega,p}(s)ds$, and by a direct calculation we have

$$\begin{aligned} F(u) &= -\frac{\omega}{2}u^2 + \frac{u^{p+1}}{p+1} - \frac{u^{2p}}{2p} \\ &= \frac{u^2}{2p(p+1)}\left[-\omega p(p+1) + 2pu^{p-1} - (p+1)u^{2(p-1)}\right]. \end{aligned} \tag{4}$$

There are two cases of concern:

(a) $\quad \omega < \omega_p \iff F$ has two zeros in $(0, \infty)$.

(b) $\quad \omega \geq \omega_p \iff F$ has at most one zero in $(0, \infty)$.

The condition to assure the existence of positive solutions of (2) given in [1, 2] is the following;

Lemma 1. *The problem* (2) *has a positive solution if and only if both of the following hypotheses are fulfilled:*

(H1) $\lim_{u \to +0} \frac{f(u)}{u}$ *exists and is negative,*

(H2) $F(\delta) > 0$ *for some positive constant* δ*.*

Lemma 2. *The problem* (2) *has a positive solution if and only if*

$$\omega \in (0, \omega_p)$$

for $p > 1$*.*

Proof. (H1) is equivalent to the condition $\omega > 0$. (H2) is equivalent to the condition (a) above. □

This is the origin of ω_p. Next we turn to the exponent a_p.
As a preparation, we calculate the derivatives of $f(u) = -\omega u + u^p - u^{2p-1}$:

$$f'(u) = -\omega + pu^{p-1} - (2p-1)u^{2(p-1)},$$
$$f''(u) = 2(p-1)(2p-1)u^{p-2}\left[\frac{p}{2(2p-1)} - u^{p-1}\right].$$

We shall introduce four positive constants α, b, c and β.

- Let α denote the unique zero of f'' in $(0, \infty)$: $\alpha = \left[\frac{p}{2(2p-1)}\right]^{\frac{1}{p-1}}$.

- Let b denote the first zero of f in $(0, \infty)$: $b = \left[\frac{1 - \sqrt{1 - 4\omega}}{2}\right]^{\frac{1}{p-1}}$.

- Let c denote the last zero of f in $(0, \infty)$: $c = \left[\frac{1 + \sqrt{1 - 4\omega}}{2}\right]^{\frac{1}{p-1}}$.

- Let β denote the first zero of F in $(0, \infty)$: $\beta = \left[\frac{p}{p+1}\left(1 - \sqrt{1 - \frac{(p+1)^2}{p}\omega}\right)\right]^{\frac{1}{p-1}}$.

It is easy to check that

$$\beta \in (b, c) \tag{5}$$

either by observing the graphs or by a straightforward calculation. From (5) we deduce

$$f(\beta) > 0, \tag{6}$$

which will be used later.

We are not able to give a clear explanation on the relation between α and β.

Lemma 3. *The condition $\alpha \leq \beta$ is equivalent to $\omega \geq a_p = \dfrac{p(7p-5)}{4(p+1)(2p-1)^2}$.*

Proof. A simple calculation. □

This is where our a_p comes into play. In the next section, we see what this condition stands for.

3 Proof of Theorem 1.

First we state the result by Peletier and Serrin [9], which assures the uniqueness of solutions of (2).

Lemma 4. *Let f satisfy (H1-3), where (H1), (H2) are in Lemma 1., and (H3) is the following:*

(H3) $G(u) := \dfrac{f(u)}{u-\beta}$ is nonincreasing in (β, c).

Then (2) has exactly one positive solution.

Now we are in a position to prove Theorem 1.

Proof of Theorem 1. We shall see that for $\omega \in [a_p, \omega_p)$, (H1-3) are satisfied. It is enough to show that if $\omega \geq a_p$, then

$$k(u) := f'(u)(u-\beta) - f(u) \leq 0 \qquad \text{in } \ (\beta, c). \tag{7}$$

To prove (7) we calculate the derivative of $k(u)$

$$k'(u) = f''(u)(u-\beta),$$

and note that

$$\begin{aligned} f''(u) &> 0 \qquad \text{in } (0, \alpha); \\ f''(u) &< 0 \qquad \text{in } (\alpha, \infty). \end{aligned}$$

So if $\alpha \leq \beta$ (i.e. $\omega \geq a_p$, see Lemma 3), then $k'(u) < 0$ in (β, c), i.e. k is decreasing in the interval. Therefore

$$k(u) < k(\beta) = -f(\beta) < 0 \qquad \text{in} \quad (\beta, c),$$

where the last inequality follows by (6).

This proves (7) and completes the proof. □

If $\alpha > \beta$, we need to check that $k(\alpha) \leq 0$, i.e.

$$\alpha - \frac{f(\alpha)}{f'(\alpha)} \leq \beta. \tag{8}$$

This condition provides an implicit relation between ω and p. Besides,

Remark 2. The condition (8) does not cover all $\omega \in (0, \omega_p)$. That is for ω close to zero, $\alpha - \dfrac{f(\alpha)}{f'(\alpha)} > \beta$.

Proof. The left hand side of (8) is estimated from below as

$$\begin{aligned} \alpha - \frac{f(\alpha)}{f'(\alpha)} &= \frac{(p-1)\alpha^p(1-2\alpha^{p-1})}{-\omega + p\alpha^{p-1} - (2p-1)\alpha^{2(p-1)}} \\ &> \frac{(p-1)\alpha^p(1-2\alpha^{p-1})}{p\alpha^{p-1} - (2p-1)\alpha^{2(p-1)}} > 0, \end{aligned}$$

for all $\omega \in (0, \omega_p)$, whereas the right hand side β decreases to zero as ω decreases to zero. $\square$

When ω is close to zero, a very delicate observation is needed. See Ouyang and Shi [8] for details.

References

[1] H. Berestycki and P. L. Lions, Nonlinear scalar field equation, I., *Arch. Rat. Math. Anal.* **82** (1983), 313-345.

[2] H. Berestycki, P. L. Lions and L. A. Peletier, An ODE approach to the existence of positive solutions for semilinear problems in $\mathbb{R}^N$, *Indiana University Math. J.* **30** (1981), 141-157.

[3] R. Fukuizumi, Stability and instability of standing waves for nonlinear Schrödinger equations, *Tohoku Mathematical Publications*, **No.25**, 2003.

[4] B. Gidas, W. M. Ni and L. Nirenberg, Symmetry and related properties via the maximal principle, *Comm. Math. Phys.* **68** (1979), 209-243.

[5] B. Gidas, W. M. Ni and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$, Mathematical analysis and applications, Part A, *Adv. in Math. Suppl. stud.* **7a**, Academic press, 1981, 369-402.

[6] M. K. Kwong and L. Zhang, Uniqueness of positive solutions of $\triangle u + f(u) = 0$ in an annulus, *Differential Integral Equations* **4** (1991), 583-599.

[7] T. Mizumachi, Uniqueness of positive solutions to a scalar field equation with a double power nonlinearity, preprint.

[8] T. Ouyang and J. Shi, Exact multiplicity of positive solutions for a class of semilinear problems, *J. Differential Equations* **146** (1998), 121-156.

[9] L. A. Peletier and J. Serrin, Uniqueness of positive solutions of semilinear equations in $\mathbb{R}^n$, *Arch. Rat. Mech. Anal.* **81** (1983), 181-197.